Journal of Pure and Applied Algebra 76 (1991) 151–153
North Holland

Extending finite mappings to affine spaces

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Communicated by M.F. Coste-Roy
Received 8 November 1990
Revised 16 January 1991

Abstract

Kwieciński, M., Extending finite mappings to affine spaces, Journal of Pure and Applied Algebra 76 (1991) 151–153.

We prove that any finite mapping of an affine algebraic set can be extended to a finite mapping of the ambient affine space.

1. Introduction

Let k be any field, V,W algebraic subsets of k^m,k^n respectively. A regular mapping $f=(f_1,\ldots,f_n):V\to W$ induces a homomorphism $f^*:k[W]\to k[V]$ defined by $f^*(p)=p(f_1,\ldots,f_n)$. A mapping f is called *finite* if $k[V]$ is a finitely generated $f^*(k[W])$-module; in other words, if $k[V]$ is an integral extension of $f^*(k[W])$. Our main goal is to prove the following theorem:

Theorem. *Let V,W be algebraic subsets of k^m,k^n respectively with $m\le n$. Let $f:V\to W$ be a finite mapping. Then there exists a finite mapping $F:k^m\to k^n$ such that $F|_V=f$.*

Remark 1. In the above theorem, the affine spaces cannot be replaced by arbitrary algebraic varieties with the same assumption on their dimensions. For example, the mapping $f:\{1\}\to\{(1,1)\}$ cannot be extended to a regular nonconstant mapping from $\mathbb{C}$ to the hyperbola $xy=1$.

Remark 2. For $k=\mathbb{C}$, the field of complex numbers, the problem of extending regular embeddings has attracted some attention and has been solved only partially—see, for example, [1, 2]. Our theorem is the solution of an extension problem in the wider class of finite mappings, which in this case are well known to be precisely those regular mappings which are proper in the Euclidean topology induced from the affine spaces (i.e., such that preimages of compact sets are compact). For an easy proof of this last characterization see, for example, [4].

2. Proof of the main theorem

Notice that we can assume $W = k^n$.

Step 1. *We construct a finite mapping* $\psi : k^m \to k^n \times k^m$, *such that* $\psi|_V = (f, 0)$.

From the finiteness of f we may choose monic polynomials

$$h_i(T) = T^{n_i} + P_{in_i-1}(f_1, \ldots, f_n)T^{n_i-1} + \cdots + P_{i0}(f_1, \ldots, f_n)$$

in one variable T over $k[f_1, \ldots, f_n]$, such that, for $i = 1, \ldots, m$ we have $h_i(x_i) = 0$ in $k[V]$, where x_i denotes the image of X_i in the natural projection $k[X_1, \ldots, X_m] \to k[V]$.

Choose any regular mapping $G = (G_1, \ldots, G_n) : k^m \to k^n$ such that $G|_V = f$ and define monic polynomials

$$H_i(T) = T^{n_i} + P_{in_i-1}(G_1, \ldots, G_n)T^{n_i-1} + \cdots + P_{i0}(G_1, \ldots, G_n).$$

The mapping

$$\psi = (G_1, \ldots, G_n, H_1(X_1), \ldots, H_m(X_m))$$

has the required properties. Indeed, $H_i(X_i) \in I(V)$ for $i = 1, \ldots, m$ and hence $\psi|_V = (f, 0)$. Furthermore, each X_i is the root of the monic polynomial $H_i(T) - H_i(X_i)$ which belongs to $k[G_1, \ldots, G_n, H_1(X_1), \ldots, H_m(X_m)][T]$, and so ψ is finite. Thus, we have completed Step 1.

Step 2. *Consider the algebraic set* U = *Zariski closure of* $\psi(k^m)$. *(In fact,* $\psi(k^m)$ *is Zariski-closed,* ψ *being finite, but we do not need that much.) We shall construct a finite mapping* $\pi : U \to k^n$ *of the form*

$$\pi = (z_1 + R_1(z_{n+1}, \ldots, z_{n+m}), \ldots, z_n + R_n(z_{n+1}, \ldots, z_{n+m})),$$

where z_i *are classes of coordinates in* $k[U]$ *and* R_i *are polynomials with coefficients in* k.

The reader is refered to Chapter II, Section 3 of [3] for basic facts about Noetherian normalization which we shall use. The following lemma, although not explicitly stated in Kunz's book, follows easily from a closer examination of Lemma 3.2 and Steps 1 and 2 of the construction used in the proof of the Noether Normalization Theorem presented therein.

Lemma. *Let $A = k[Z_1, \ldots, Z_s]/I$, where I is a proper ideal of the polynomial ring $k[Z_1, \ldots, Z_s]$. For any natural number n, such that $\dim A \le n \le s$, there exist polynomials $R_1(Z_{n+1}, \ldots, Z_s), \ldots, R_n(Z_{n+1}, \ldots, Z_s)$ such that A is a finitely generated $k[Z_1 + R_1(Z_{n+1}, \ldots, Z_s), \ldots, Z_n + R_n(Z_{n+1}, \ldots, Z_s)]$-module* (dim *denotes Krull dimension). If the field k is infinite, then we may choose R_i linear.* □

Let us now proceed to construct the mapping π. First, notice that the Krull dimension of $k[U]$ is m. Indeed, if $k[Y_1, \ldots, Y_d]$ is a Noetherian normalization of $k[U]$, then, by the finiteness of ψ, $k[\psi^*(Y_1), \ldots, \psi^*(Y_d)]$ is a Noetherian normalization of $k[X_1, \ldots, X_m]$, and hence obviously $\dim k[U] = d = \dim k[X_1, \ldots, X_m] = m$.

Hence, since $n \ge m = \dim k[U]$ we can apply the lemma to the k-algebra $k[U]$. The mapping π defined with polynomials R_i from the lemma (as at the beginning of this step) has the required properties. This finishes Step 2.

Step 3. *Construction of the mapping F.*

The mapping $F = \tau \circ \pi \circ \psi$, where τ is the translation of k^n by the vector $(-R_1(0, \ldots, 0), \ldots, -R_n(0, \ldots, 0))$ is finite (as the composition of finite mappings) and obviously $F|_V = f$. This finishes Step 3 and the proof of the main theorem. □

Acknowledgment

The author would like to thank Tadeusz Winiarski for stating the problem and Kamil Rusek and Krzysztof Kurdyka for their helpful comments on this paper.

References

[1] S.S. Abhyankar, On the semigroup of a meromorphic curve (Kinokuniya Book Store, Tokyo, 1978).

[2] Z. Jelonek, The extension of regular and rational embeddings, Math. Ann. 277 (1987) 113–120.

[3] E. Kunz, Einführung in die Kommutative Algebra und Algebraische Geometrie (Vieweg, Braunschweig, Wiesbaden, 1979).

[4] K.J Nowak, The extension of holomorphic functions of polynomial growth on algebraic sets in $\mathbb{C}^n$, Univ. Iagel. Acta Math., to appear; Osnabrück. Schrift. Math. 75 (1984) (preprint).